Indian Statistical Institute, Bangalore Centre
B.Math (Hons.) II Year, First Semester
Semestral Examination - 2013-2014
Analysis III

Time: 3 Hours November 6, 2013

Maximum marks you can get is 50. This paper carries 54 marks.

1. (a) Let $f, h : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be $\mathcal{C}^1$ functions. Let V be the convex closed tetrahedron of R^3 with vertices $0 = (0,0,0),\ A = (a,0,0),\ B = (0,b,0)$ and $C = (0,0,c)$ where $a > 0,\ b > 0,\ c > 0$. Let $\underset{\sim}{F} = (f,0,0)$ Or $(0,0,h)$. Let S be the boundary of V, with outward normal orientation $\underset{\sim}{n}$. Show that $\iint\limits_S \underset{\sim}{F} = \iiint\limits_V \text{div } \underset{\sim}{F}$. [6]

   (b) Let $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be any $\mathcal{C}^1$ function. Let S be any bounded smooth surface in R^3 with $\underset{\sim}{n}$, unit normal $\mathcal{C}^1$ family for S. Let $\underset{\sim}{F} = h\underset{\sim}{n}$ on S. Define $\iint\limits_{(S,\underset{\sim}{n})} h = \iint\limits_S h\underset{\sim}{n}$. Let $\underset{\sim}{\psi}$ represent S i.e., G is an open set in $R^2, \underset{\sim}{\psi} : G \longrightarrow R^3$ is $\mathcal{C}^1, \underset{\sim}{\psi}$ is $1-1$ and $\underset{\sim}{\psi}(G) = S$. Show that $\int\limits_{(S,\underset{\sim}{n})}\int h = \int\limits_G\int h(\underset{\sim}{\psi}(u,v))\ ||\ \underset{\sim}{\psi}_u \times \underset{\sim}{\psi}_v\ ||\ du\, dv$ [1]

   (c) Let $f, g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be $\mathcal{C}^2$ functions. Let $V, S, \underset{\sim}{n}$ be as in previous parts. Define $h : R^3 \longrightarrow R$ by $h = f\frac{\partial g}{\partial \underset{\sim}{n}} - g\frac{\partial f}{\partial \underset{\sim}{n}}$. Let Δ be the Laplacian on R^3 given by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Show that $\iint\limits_{(S,\underset{\sim}{n})} h = \iiint\limits_V (f\ \Delta\ g - g\ \Delta\ f)$. [10]

2. (a) Two $\mathcal{C}^2$ functions $u, v : \Omega \longrightarrow R$, where Ω an open set in R^2 are said to be harmonic conjugates if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Let C be any circle given by $C = \{(x-a)^2 + (y-b)^2 = r^2\}$. Assume that $\{(x-a)^2 + (y-b)^2 \leq r^2\} \subset \Omega$. Show that $\int\limits_C udx - vdy = 0 = \int\limits_C vdx + udy$ [2]

   (b) Let $f, g : R^2 \longrightarrow R$ be given by $f(x,y) = e^x \cos y, g(x,y) = e^x \sin y$ with C as in part (a), find $\int\limits_C fdx - gdy$ and $\int\limits_C gdx + fdy$ [2]

   (c) Let $P, Q : R^2 - \{(0,0)\} \longrightarrow R$ be given by $P(x,y) = \frac{-y}{x^2+y^2}, Q(x,y) = \frac{x}{x^2+y^2}$
   Show that
   (i) $\int\limits_{x^2+y^2=1} Pdx + Qdy \neq 0$ [1]
   (ii) Show that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ on $R^2 - \{(0,0)\}$ [1]
   (iii) Is there a contradiction between Greens theorem and (i)+(ii)?. Justify your answer. [2]

3. Let $\underset{\sim}{F}(x,y,z) = (y^2, xy, xz)$. Let S be the portion of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ above the plane $z = \frac{c}{2}$. Here $a > 0,\ b > 0,\ c > 0$. Find $\iint\limits_S \text{curl } \underset{\sim}{F}$. [4]

4. Calculate $\iint\limits_{H} e^{x+y}\, dx\, dy$

where H is the parallelogram with vertices at $(1,1),(3,2),(4,5),(2,4)$ [5]

5. Let $n > 2, a_1 > 0,\ a_2 > 0, \cdots a_n > 0$. Let

$$S(a_1, a_2, \cdots a_n) = \{(y_1, y_2, \cdots y_n) : |\frac{y_i}{a_i}| + |\frac{y_n}{a_n}| \leq 1 \text{ for each } i = 1, 2, \cdots n-1\}$$

Calculate $\int\limits_{S(a_1,\ a_2,\ \cdots a_n)} dy$ [5]

6. (a) Let $f_1, f_2, \cdots, f_n, \cdots$ be a sequence of real valued $\mathcal{C}^1$ functions on $[a,b]$. Let $h, g : [a,b] \longrightarrow R$ be continuous functions. If $f_n \longrightarrow g$ and $f_n' \longrightarrow h$ uniformly on $[a,b]$, then show that g is differentiable and $g' = h$. [3]

   (b) Let $f(t,x) = \sum\limits_{n=1}^{\infty} a_n\ e^{-nt}\ \sin(nx)$ for $t > 0,\ x$ in R. Here $a_1, a_2, \cdots$ is a bounded sequence of real numbers. Formally you can verify that

$$\frac{\partial^2 f}{\partial t^2}\ (t,x) = -\frac{\partial^2}{\partial x^2}\ f(t,x)$$

   Prove it rigorously by stating a theorem similar to Weirstrass M-test. [7]

7. (a) State implicit function theorem. [2]

   (b) Let $f : R^2 \longrightarrow R$ be given by $f(x,y) = x^2+y^2$ and $C = \{f = 1\}$. Clearly$(1,0) \in C$. Show that there does not exist $\delta > 0$ and a $\mathcal{C}^1$ function $g : (1-\delta,\ 1+\delta) \longrightarrow R$ such that $g(1) = 0, (x, g(x)) \in C$ for all x in $(1-\delta,\ 1+\delta)$. [2]

   (c) Is there a contradiction between (a) and (b)?. Justify your answer. [1]